# Solutions

## 4.3: Counting with Functions

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**Question 1.** A *quaternary string* is a string made up of 0's, 1's, 2's, and 3's. Devise a method for converting a quaternary string of length $n$ to a binary string of length $2n$, and vice versa. (Start with the case of $n = 2$.) If possible, ensure that your conversion methods return the original string when composed.

Let $a = a_1 a_2 \cdots a_n$ be a quaternary string of length $n$.

For each digit $a_k$, we want to map

$$\begin{aligned} 0 &\mapsto 00 \\ 1 &\mapsto 01 \\ 2 &\mapsto 10 \\ 3 &\mapsto 11. \end{aligned}$$

Call this function $f : \mathbb{Z}/4 \to \mathbb{Z}/2 \times \mathbb{Z}/2$.

Then extending $f$ to $f_n : (\mathbb{Z}/4)^n \to (\mathbb{Z}/2 \times \mathbb{Z}/2)^n$, we have

$$f_n(a) = f(a_1) f(a_2) \cdots f(a_n).$$

We reverse this process by defining $g : \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}_4$ by

$$\begin{aligned} 00 &\mapsto 0 \\ 01 &\mapsto 1 \\ 10 &\mapsto 2 \\ 11 &\mapsto 3. \end{aligned}$$

It is easy to see that $g = f^{-1}$.

Therefore, $g_n : (\mathbb{Z}/2 \times \mathbb{Z}/2)^n \to \mathbb{Z}_4^n$ given by

$$g_n(b) = g(b_1 b_2) g(b_3 b_4) \cdots g(b_{2n-1} b_{2n})$$

is the inverse of $f_n$.

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For the case $n=2$, we can see that

$$g_2(f_2(00)) = g_2(0000) = 00$$

$$g_2(f_2(01)) = g_2(0001) = 01$$

$$g_2(f_2(02)) = g_2(0010) = 02$$

etc.

Therefore this procedure does indeed recover the original string.

## One-to-One Correspondences.

**Theorem 1.** *Let $|X| = m$ and $|Y| = n$. If there is some function $f : X \to Y$ that is one-to-one, then $m \leq n$.*

**Theorem 2.** *Let $|X| = m$ and $|Y| = n$. If there is some function $f : X \to Y$ that is onto, then $m \geq n$.*

**Corollary 3.** *Let $|X| = m$ and $|Y| = n$. If there is a one-to-one correspondence $f : X \to Y$, then $m = n$.*

**Example 1.** In a single-elimination tournament, players are paired up in each round, and the winner of each match advances to the next round. If the number of players in a round is odd, one player gets a bye to the next round. The tournament continues until only two players are left; these two players play the championship game to determine the winner of the tournament. In a tournament of 270 players, how many games must be played?

Round 1: $\frac{270}{2} = 135$ games

Round 2: $\frac{135}{2} = 67.5 \Rightarrow 67$ games

Round 3: $\frac{68}{2} = 34$ games

Round 4: $\frac{34}{2} = 17$ games

Round 5: $\frac{17}{2} = 8.5 \Rightarrow 8$ games

Round 6: $\frac{9}{2} = 4.5 \Rightarrow 4$ games

Round 7: $\frac{5}{2} = 2.5 \Rightarrow 2$ games

Round 8: $\frac{3}{2} = 1.5 \Rightarrow 1$ game

Round 9: $\frac{2}{1} = 1$ game

Total Games $= 135 + 67 + 34 + 17 + 8 + 4 + 2 + 1 + 1$

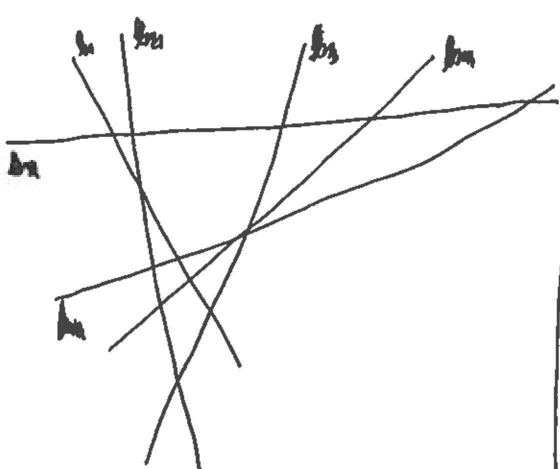
$= 269$ games

$= |G|$.

By noticing that all but one player must lose exactly one game, we can define a function from $L$ = the set of all losers to $G$ = the set of all games. Specifically, $f : L \to G$ is defined by $f(l) = g$ if player $l$ lost in game $g$. Since this function is a one-to-one correspondence, we know that $|G| = |L| = 269$ by Cor. 3.

**Example 2.** Six lines are drawn such that every line intersects every other line and no three lines intersect in a single point. How many triangles are formed?

For example:



~~Since~~ Notice that

$$\{\text{triangles in the figure}\} \longleftrightarrow \{\text{sets } \{l_1, l_2, l_3\} \mid l_i \text{ is a line}\},$$

that is to say the triangles in the figure are in a one-to-one corresponce with the subsets of 3 lines from the figure or, in other words, each triangle corresponds to exactly one collection of 3 lines.

Therefore the number of triangles is equal to the number of collections of 3 lines from the total of 6 lines. which is given by

$$C(6,3) = 20.$$

**Definition 4.** *A function $f : X \to Y$ is called $n - to - one$ if every $y$ in the image of the function has exactly $n$ different elements of $X$ that map to it. In other words, $f$ is $n$-to-one if*

$$|\{x \in X | f(x) = y\}| = n \quad \text{for all } y \in Y.$$

**Example 3.** Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{0, 1\}$. Define a function $m : X \to Y$ by

$$m(x) = x \mod 2.$$

This function is 3-to-one.

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**Theorem 5.** *Let $|X| = p$ and $|Y| = q$. If there is an $n$-to-one function $f : X \to Y$ that maps $X$ onto $Y$, then $p = qn$.*

**Example 4.** Prove that $P(n, r) = r! \cdot C(n, r)$.

Let $X$ be a set with $|X|=n$.

<u>Proof:</u> Recall that $P(n,r) = |\{(x_1,x_2,\cdots,x_r) \in X^r \mid x_i \neq x_j \text{ for all } i \neq j\}|$.

Let us call this set $A$ = the set of all <u>arangements</u> of $r$ elements from the set $X$.

Also recall that $C(n,r) = |\{\{x_1,x_2,\cdots,x_r\} \subseteq X \mid |\{x_1,x_2,\cdots,x_r\}| = r\}|$.

Let us call this set $S$ = the set of all <u>selections</u> of $r$ elements from the set $X$.

Define the map $f : A \to S$ by

$$f((x_1,x_2,\cdots,x_r)) = \{x_1,x_2,\cdots,x_r\} \quad \text{for every } (x_1,x_2,\cdots,x_r) \in A.$$

Clearly $f$ is onto.

Furthermore, for each set $\{x_1,x_2,\cdots,x_r\}$ in $S$, the range of $f$, we can see that

$$|\{a \in A \mid f(a) = \{x_1,x_2,\cdots,x_r\}\}| = P(r,r);$$

i.e. the number of ways to order the set $\{x_1,x_2,\cdots,x_r\}$ is $P(r,r) = r!$.

Therefore $f$ is an $r!$-to-one function and Theorem 5 tells us that

$$P(n,r) = |A| = r!\,|S| = r! \cdot C(n,r) \text{ as desired.}$$

**Example 5.** How many different strings can you form by rearranging the letters in the word ENUMERATE?

If there were no multiple letters this would be simply $P(9,9) = 9!$.
However there are repeated letters. Using methods of the previous section we can see that there are

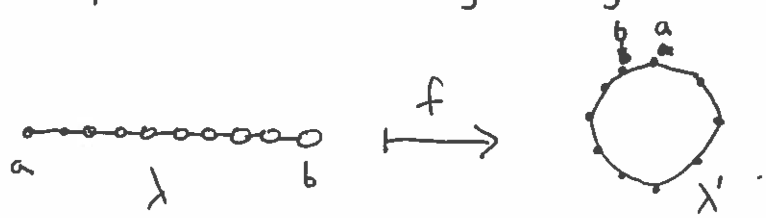
$$\underbrace{C(9,3)}_{\text{Place the 3 E's}} \cdot \underbrace{P(6,6)}_{\text{Place the remaining non-repeated letters}} = \frac{9!}{6!3!} \cdot \frac{6!}{0!} = \frac{9!}{3!} = 60{,}480 \text{ ways.}$$

We can do this using a 6-to-one function by labeling the 3 E's as $E_1, E_2$ and $E_3$ and mapping $f$ an arrangement of $E_1NUME_2RATE_3$ to an arrangement of ENUMERATE without labelings. For instance $f(NUE_3ME_1TARE_2) = NUEMETARE$.
Then the number~~ing~~ of rearrangements is equal to

$$P(9,9)/6 = \frac{9!}{6} = 60{,}480 \text{ ways.}$$

**Example 6.** A group of 10 people sit in a circle around a campfire. How many different seating arrangements are there? Let us agree that a seating arrangement is determined only by the neighbors of each person, not by where on the ground they sit or the orientation of the circle.

Let us map a row seating arrangement to a circle seating arrangement by




Call this a function $f: X \to Y$. Clearly $f$ is onto.
We know that $|X| = P(10,10) = 10!$.

Noticing that $f(\lambda) = f(\lambda^R)$, where $\lambda^R$ is the reverse of the string $\lambda$ introduced in Section 3.3, and that there are 10 places to break the circle, we see that $f$ is a 20-to-one function.

Hence $|Y| = |X|/20 = 10!/20 = 181{,}440$ circular seating arrangements.

## The Pigeonhole Principle.

**Theorem 6.** *Let $|X| = n$ and $|C| = r$, and let $f : X \to C$. If $n > r$, then there are distinct elements $x, y \in X$ with $f(x) = f(y)$.*

**Example 7.** In a club with 400 members, must there be some pair of members who share the same birthday?

Yes. There are only 366 (possible) birthdays and 400 people.
Let $B =$ possible birthdays $= \{\text{Jan } 1, \dots, \text{Dec. } 31\}$ and $P =$ the 400 people.
Then $f : P \to C$ satisfies the conditions of the Pigeonhole Principle and therefore
$f(p_1) = f(p_2) =$ same birthday for some two people $p_1$ and $p_2$.

**Example 8.** Chandra has a drawer with 12 red and 14 green socks. He must grab a selection of clothes in the dark to avoid waking his roommate. How many socks must he grab to be assured of having a matching pair?

He must grab 3. Suppose $f(s) =$ color of the sock $s$.
By the Pigeonhole Principle, if you have 3 socks then two must be the same color.

**Example 9.** In a round-robin tournament, every player plays every other player exactly once. Prove that, if no player goes undefeated, at the end of the tournament there must be two players with the same number of wins.

Suppose there are $n$-players in the round-robin tournament.
Then each player plays $n-1$ games and therefore they could win anywhere from 0 to $n-1$ games.
That is, if $w(p) =$ # of wins by player $p$, then $w(p) \in \{0, 1, \dots, n-1\}$.
Notice that $|\{0, 1, \dots, n-1\}| = n$. Supposing no player wins all games, then $w(p) \in \{0, 1, \dots, n-2\}$ which has size $|\{0, 1, \dots, n-2\}| = n-1$.
Since there are $n$ players and only $n-1$ possible # of wins, 2 player must have the same # of wins by the Pigeonhole Principle.

## The Generalized Pigeonhole Principle.

**Theorem 7.** *Let $|X| = n$ and $|C| = r$, and let $f : X \to C$. If $n > r(l-1)$, then there is some subset $U \subseteq X$ such that $|U| = l$ and $f(x) = f(y)$ for all $x, y \in U$.*

**Corollary 8.** *Let $|X| = n$ and $|C| = r$, and let $f : X \to C$. Then there is some subset $U \subseteq X$ such that*

$$|U| = \left\lceil \frac{n}{r} \right\rceil$$

*and $f(x) = f(y)$ for any $x, y \in U$.*

**Example 10.** A website displays an image each day from a bank of 30 images. In any given 100-day period, show that some image must be displayed four times.

Since $\left\lceil \frac{100}{30} \right\rceil = 4$, the result follows by the Generalized PHP. (Technically Corollary 8).

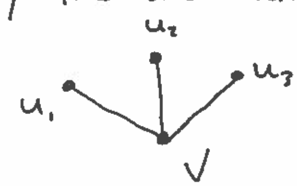
**Example 11.** Let $G$ be a complete graph on six vertices. Suppose the 15 edges of this graph are colored red or green. Show that there must be some triangular circuit whose edges are the same color.

The complete graph on n vertices has $C(6,2) = 15$ edges.

~~It also contains C(6,3) = 20 triangles.~~

Pick a vertex v. There are 5 edges on v. By ~~the~~ the Generalized PHP 3 of the must be the same color.

Suppose these edges are green without loss of generality.




Case 1: At least one of the edges connecting $u_1, u_2$ and $u_3$ is green. Suppose it is $u_1$ and $u_2$. Then $\{u_1, u_2, v\}$ forms a triangle with all green edges.

Case 2: No edges are green. Then they are all red and $\{u_1, u_2, u_3\}$ forms a triangle with all red edges.

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**No Homework.**
**Practice Problems.** Section 4.3: 3-13, 18-23, 28, 30